Assignment 9

Deadline: March 28, 2018

Hand in no 1, 2, 4, 7.

Supplementary Exercise

1. Use the Weierstrass M-test to study the uniform convergence of the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$ for $x \in (0, b)$ where $b > 0$.

2. Show that the series $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^p}$ defines a continuous function on $\mathbb{R}$ for $p > 1$.

3. Show that the infinite series $\sum_{j=1}^{\infty} \dfrac{\cos 2^j x}{3^j}$ is a continuous function on the real line. Is it differentiable?

4. Show that the sequence $g_n(x) = \sum_{j=0}^{n} e^{-jx}$ defines a smooth function on $[1, \infty)$. What will happen if $[1, \infty)$ is replaced by $[0, \infty)$?

5. (a) Suppose that $\sum_{n=1}^{\infty} f_n(x)$ is pointwisely convergent on E and g is a function on E. Show that $\sum_{n=1}^{\infty} g(x) f_n(x)$ pointwisely converges to $g(x) \sum_{n=1}^{\infty} f_n(x)$, that is,

$$\sum_{n=1}^{\infty} g(x) f_n(x) = g(x) \sum_{n=1}^{\infty} f_n(x) \ .$$

   (b) Suppose further that $\sum_n f_n$ converges uniformly and g is bounded, show that $\sum_n g f_n$ converges uniformly.

6. Suppose f is a nonzero function satisfying $f(x + y) = f(x) f(y)$ for all real numbers x and y and is differentiable at $x = 0$. Show that it must be of the form e^{ax} for some number a. Hint: Study the differential equation f satisfies. Show that $f(0) = 1$ first.

7. (a) Show that

$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} \le E(x) \le 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^a x^n}{n!} \ , \quad x \in [0, a] \ .$$

   (b) Show that e is not a rational number. Suggestion: Deduce from (a) the inequality

$$0 < en! - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) n! < \frac{e}{n+1} \ .$$

8. Show that the series

$$\sum_{j=0}^{\infty} \frac{x^j}{j!}$$

is not uniformly convergent on $\mathbb{R}$ (although it is uniformly convergent in every $[-M, M]$).

9. Optional. Let a be a positive number and $n \in \mathbb{N}$.
   (a) Show that there is a unique positive number b satisfying $b^n = a$. Write $b = a^{1/n}$.
   (b) For any rational number $m/n, m \in \mathbb{Z}, n \in \mathbb{N}$, define $a^{m/n} = (a^m)^{1/n}$. Show that $a^{m/n} = (a^{1/n})^m$.
   (c) Show that $a^{r_1+r_2} = a^{r_1}a^{r_2}$ for rational numbers r_1, r_2.

   This is 2050 stuff. It serves to refresh your memory.